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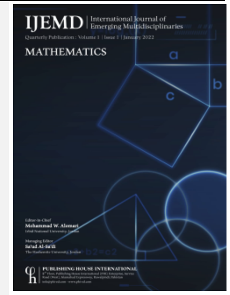

Approximate Solution of Generalized Modified *b*-Equation by Optimal Auxiliary Function Method

Rashid Nawaz[1], Laiq Zada[1] and Aatif Ali[1*]

[1] Department of Mathematics, Abdul Wali Khan University Mardan, 23200, KP, Pakistan
**Corresponding author*

Abstract

In this study, the implantation of a new semi-analytical method called the optimal auxiliary function method (OAFM) has been extended to partial differential equations. The adopted method was tested upon for approximate solution of generalized modified b-equation. The first-order numerical solution obtained by OAFM has been compared with the variational homotopy perturbation method (VHPM). The method possesses the auxiliary function and control parameters which can be easily handled during simulation of the nonlinear problem. From the numerical and graphical results, we concluded the method is very effective and easy to implement for the nonlinear PDEs.

Keywords: Approximate solution, Modified b-Equation, Optimal Auxiliary Function Method (OAFM).
2010 Mathematics Subject Classification: 35C10; 35A24; 35R11

1. Introduction

Differential equations (DE) play a vital role in applied science and engineering. PDEs have a variety of applications in optics, hydrodynamics electromagnetism, economics, financial mathematics, and computer science. Usually, nonlinear PDEs don't have the exact solutions, therefore the researchers adopted different approaches for the approximate solution. In such difficult cases, it's so much tough to obtain the exact solution of these nonlinear differential equations. Therefore, a different method has been used in literature for these types of equations [1-5].

In the same research field, we extend the implementation of the OAFM partial differential equations and applied for the approximate solution of modified b-equation. This planed method was introduced by Marinca et.al and used for the solution for the fluid model [6]. Later on, the proposed method was used by researchers for different problems in the field of applied mathematics [7-10]. We aim to apply the proposed method to the generalized modified b –equation. The proposed model was introduced by Wazwaz [11] to study a family of

Email addresses: rashid_uop@yahoo.com (R.Nawaz), laiq0023@gmail.com (Laiq Zada), atifkh98@gmail.com (Aatif Ali)

the important physical equation, namely called modified b–equation. The modified b-equation is given as follow,

$$\frac{\partial\zeta(\eta,t)}{\partial\tau}-\frac{\partial^3\zeta(\eta,t)}{\partial\eta^3\tau}+\left(b+1\right)\zeta^2(\eta,t)\frac{\partial\zeta(\eta,t)}{\partial\eta}-b\frac{\partial\zeta(\eta,t)}{\partial\eta}\frac{\partial^2\zeta(\eta,t)}{\partial\eta^2}-\frac{\partial\zeta(\eta,t)}{\partial\eta}\frac{\partial^2\zeta(\eta,t)}{\partial\eta^2}=0, \tag{1}$$

In eq. (1) b is a positive integer. For b =3, eq. (1) is reduced to modified camassa holom (mCH) equation, while, for b=4 eq. (1) reduced to modified Degpress (mDP) equation. The modified b-equation has been studied by different methods in the series of papers [12-14].

2. OAFM Methodology for PDEs

The extended for of OAFM for partial differential equation has been discussed in the following steps. Let

$$L\left[\zeta(\eta,\tau)\right]=f(\eta,\tau)+N\left[\zeta(\eta,\tau)\right]=0, \tag{2}$$

subject to the boundary condition

$$\mathrm{B}\left(\zeta,\frac{\partial\zeta}{\partial\tau}\right). \tag{3}$$

Hence L , f and N presents a linear operator, a known function, and a nonlinear operator respectively.

Step1: To find the approximate solution of Eq. (2), Let the approximate solution can be expressed in the form of two components given,

$$\tilde{\zeta}(\eta,\tau)=\zeta_0(\eta,\tau)+\zeta_1(\eta,\tau,C_i),\quad i=1,2,......\mathrm{P} \tag{4}$$

Step 2: To find the initial and the first approximation, we substitute Eq. (4) into Eq. (2), it results in

$$L\left[\zeta_0(\eta,\tau)\right]+L\left[\zeta_1(\eta,\tau,C_i)\right]+f(\eta,\tau)+N\left[\zeta_0(\eta,\tau)+\zeta_1(\eta,\tau,C_i)\right]=0. \tag{5}$$

Step 3: To obtain $\zeta_0(\eta,\tau)$ and first order solution $\zeta_1(\eta,\tau)$, we consider the following linear equations:

$$L\left[\zeta_0(\eta,\tau)+f(\eta,\tau)\right]=0,\quad \mathrm{B}\left(\zeta_0,\frac{\partial\zeta_0}{\partial\tau}\right)=0, \tag{6}$$

$$L\left[\zeta_1(\eta,\tau,C_i)\right]+N\left[\zeta_0(\eta,\tau)+\zeta_1(\eta,\tau,C_i)\right]=0,\quad \mathrm{B}\left(\zeta_1,\frac{\partial\zeta_1}{\partial\tau}\right)=0. \tag{7}$$

Step 4: The nonlinear term from eq. (7) is expanded in the form of

$$N\left[\zeta_0(\eta,\tau)+\zeta_1(\eta,\tau,C_i)\right]=N\left[\zeta_0(\eta,\tau)\right]+\sum_{k=1}^{\infty}\frac{\zeta_1^k}{k!}N^{(k)}\left[\zeta_0(\eta,\tau)\right]. \tag{8}$$

Step 5: Equation (7) is very difficult to solve, so we propose another expression for controlling and accelerating the convergence of the method. So, eq. (7) can be written as

$$\begin{gathered}L\left[\zeta_1(\eta,\tau,C_i)\right]+\mathrm{A}_1\left[\zeta_0(\eta,\tau)\right]N\left[\zeta_0(\eta,\tau)\right]+\mathrm{A}_2\left[\zeta_0(\eta,\tau),C_j\right]=0,\\ \mathrm{B}\left(\zeta_1,\frac{\partial\zeta_1}{\partial\tau}\right)=0,\end{gathered} \tag{9}$$

Remark 1: In Eq.(9), A_1 and A_2 are known auxiliary functions, that can be chosen based on initial guesses and unknown parameters C_i and C_j , $i=1,2,3...,\quad j=s+1,s+2,3,..\mathrm{p}$.

Remark 2: A_1 and A_2 are not unique and are of the same form like $\zeta_0(\eta,\tau)$ are the form of $N\left[\zeta_0(\eta,\tau)\right]$ or the combination of both $\zeta_0(\eta,\tau)$ and $N\left[\zeta_0(\eta,\tau)\right]$.

Remark 3:

(1). If $\zeta_0(\eta,\tau)$ or $N\left[\zeta_0(\eta,\tau)\right]$ a polynomial function, exponential function, or trigonometric function then the corresponding auxiliary functions should be the sum of polynomial, exponential or trigonometric functions.

(2). If in special case $N\left[\zeta_0(\eta,\tau)\right]=0$ then it is clear that $\zeta_0(\eta,\tau)$ is an exact solution of Eq. (2).

Step 6: For calculating the C_i and C_j, we use the method of least square, by minimizing the square residual error:

$$J\left(C_i,C_j\right)=\int_0^t\int_\Omega R^2\left(x,t;C_i,C_j\right)dxdt, \tag{10}$$

Hence, R is the residual i.e

$$R\left(\eta,\tau,C_i,C_j\right)=L\left[\tilde{\zeta}\left(\eta,\tau,C_i,C_j\right)\right]+f(\eta,\tau)+N\left[\tilde{\zeta}\left(\eta,\tau,C_i,C_j\right)\right],\ i=1,2,..s,\ j=S+1,S+2,..p, \tag{11}$$

3. Applications of the Method

In this section, the adopted method is used for the numerical solution of the mCH and DP equation. Additionally, we used the Mathematica 11.0 and Math type for huge computational work.

3.1. Modified Camassa-Holm Equation (mCH)

1st consider Modified Camassa-Holm Equation with initial condition and Exact solution as:

$$\frac{\partial\zeta(\eta,\tau)}{\partial\tau}-\frac{\partial^3\zeta(\eta,\tau)}{\partial\eta^2\partial\tau}+3\zeta^2\frac{\partial\zeta(\eta,\tau)}{\partial\eta}-2\frac{\partial\zeta(\eta,\tau)}{\partial\eta}\frac{\partial^2\zeta(\eta,\tau)}{\partial\eta^2}-\zeta\frac{\partial^2\zeta(\eta,\tau)}{\partial\eta^2}=0, \tag{12}$$

subject to initial condition

$$\zeta(\eta,0)=-2\operatorname{sech}^2\left(\frac{1}{2}\eta\right). \tag{13}$$

In eq. (12) linear and nonlinear are given as

$$L[\zeta(\eta,\tau)]=\frac{\partial\zeta(\eta,\tau)}{\partial\tau}. \tag{14}$$

$$f[(\eta,\tau)]=0. \tag{15}$$

$$N\left[\zeta\right]=-\frac{\partial^3\zeta}{\partial x^2\partial t}+3\zeta^2\frac{\partial\zeta}{\partial x}-2\frac{\partial\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x^2}-\zeta\frac{\partial^2\zeta}{\partial x^2}. \tag{16}$$

The initial approximate $\zeta_0(\eta,\tau)$ is obtained from eq. (6).

$$\frac{\partial\zeta_0(\eta,\tau)}{\partial\tau}=0,\quad \zeta_0(\eta,0)=-2\operatorname{sech}^2\left(\frac{1}{2}\eta\right). \tag{17}$$

The solution of eq. (4.4) is

$$\zeta_0(\eta,\tau) = -2\,\text{sech}^2\left(\frac{1}{2}\eta\right). \tag{18}$$

use eq. (18) into eq. (16), the nonlinear operator becomes

$$\begin{aligned}N\left[\zeta_0(\eta,\tau)\right] &= 24\,\text{sech}^6\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right) - 4\,\text{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)\left(\text{sech}^4\left(\frac{\eta}{2}\right) - 2\,\text{sech}^2\left(\frac{\eta}{2}\right)\tanh^2\left(\frac{\eta}{2}\right)\right)\\ &+2\,\text{sech}^2\left(\frac{\eta}{2}\right)\left(-4\,\text{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right) + 2\,\text{sech}^2\left(\frac{\eta}{2}\right)\tanh^3\left(\frac{\eta}{2}\right)\right).\end{aligned} \tag{19}$$

The first approximation $\zeta_1(\eta,\tau)$ is given by eq. (9)

$$\begin{aligned}&\frac{\partial \zeta_1(\eta,\tau)}{\partial \tau} + A_1\left[\zeta_0(\eta,\tau)\right]N\left[\zeta_0(\eta,\tau)\right] + A_2\left[\zeta_0(\eta,\tau), C_j\right] = 0,\\ &\zeta_1(\eta,0) = 0,\end{aligned} \tag{20}$$

Here we choose A_1 and A_2 as

$$\begin{cases} A_1 = C_1\left[\text{sech}\left(\frac{\eta}{2}\right)\right]^2 + C_2\left[\text{sech}\left(\frac{\eta}{2}\right)\right]^4 \\ A_2 = C_3\left[\text{sech}\left(\frac{\eta}{2}\right)\right]^6 + C_4\left[\text{sech}\left(\frac{\eta}{2}\right)\right]^8. \end{cases} \tag{21}$$

Using eq. (19), and (21) into Eq. (18), we get the first approximation as

$$\begin{aligned}\zeta_1(\eta,\tau) &= -\tau C_3\,\text{sech}^6\left(\frac{\eta}{2}\right) + \tau C_4\,\text{sech}^8\left(\frac{\eta}{2}\right) + 12\tau C_1\,\text{sech}^6\left(\frac{\eta}{2}\right)\text{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)\\ &+12\tau C_2\,\text{sech}^6\left(\frac{\eta}{2}\right)\text{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right) + 12\tau C_1\,\text{sech}^4\left(\frac{\eta}{2}\right)\text{sech}^2\left(\frac{\eta}{2}\right)\tanh^3\left(\frac{\eta}{2}\right).\end{aligned} \tag{22}$$

Adding eq. (18) and eq. (22) we get 1st order approximate solution as

$$\begin{aligned}\tilde{\zeta}(\eta,\tau) &= -2\,\text{sech}^2\left(\frac{1}{2}\eta\right) + -\tau C_3\,\text{sech}^6\left(\frac{\eta}{2}\right) + \tau C_4\,\text{sech}^8\left(\frac{\eta}{2}\right) + 12\tau C_1\,\text{sech}^6\left(\frac{\eta}{2}\right)\text{sech}^2\left(\frac{\eta}{2}\right)\\ &\tanh\left(\frac{\eta}{2}\right) + 12\tau C_2\,\text{sech}^6\left(\frac{\eta}{2}\right)\text{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right) + 12\tau C_1\,\text{sech}^4\left(\frac{\eta}{2}\right)\text{sech}^2\left(\frac{\eta}{2}\right)\tanh^3\left(\frac{\eta}{2}\right).\end{aligned} \tag{23}$$

3.2. Modified Degasperis-Procesi (mDP)

First consider Modified Degasperis-Procesi with initial condition and exact solution as:

$$\frac{\partial \zeta(\eta,\tau)}{\partial \tau} - \frac{\partial^3 \zeta(\eta,\tau)}{\partial \eta^2 \partial \tau} + 4\zeta^2\frac{\partial \zeta(\eta,\tau)}{\partial \eta} - 3\frac{\partial \zeta(\eta,\tau)}{\partial \eta}\frac{\partial^2 \zeta(\eta,\tau)}{\partial \eta^2} - \zeta\frac{\partial^2 \zeta(\eta,\tau)}{\partial \eta^2} = 0, \tag{24}$$

subject to initial condition,

$$\zeta(\eta,0) = -\frac{15}{8}\text{sech}^2\left(\frac{1}{2}\eta\right). \tag{25}$$

Terms to be consider in eq. (24) as,

$$L\left[\zeta(\eta,\tau)\right]=\frac{\partial\zeta(\eta,\tau)}{\partial\tau}, \tag{26}$$

$$N\left[\zeta(\eta,\tau)\right]=-\frac{\partial^3\zeta}{\partial\eta^2\partial\tau}+4\zeta^2\frac{\partial\zeta}{\partial\eta}-3\frac{\partial\zeta}{\partial\eta}\frac{\partial^2\zeta}{\partial\eta^2}-\zeta\frac{\partial^2\zeta}{\partial\eta^2}, \tag{27}$$

The initial approximate $\zeta_0(\eta,\tau)$ is obtained from eq. (6)

$$\frac{\partial\zeta_0(\eta,t)}{\partial\tau}=0,\quad \zeta_0(\eta,0)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}\eta\right). \tag{28}$$

The solution of eq. (28) is

$$\zeta_0(\eta,t)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}\eta\right). \tag{29}$$

use eq. (29) into eq. (27), the nonlinear operator becomes,

$$\begin{aligned} N\left[\zeta_0(\eta,t)\right]=&\frac{3375}{128}\operatorname{sech}^6\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)-\frac{45}{8}\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)\\ &\left(\frac{15}{16}\operatorname{sech}^4\left(\frac{\eta}{2}\right)-\frac{15}{8}\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh^2\left(\frac{\eta}{2}\right)\right)+\frac{15}{8}\operatorname{sech}^2\left(\frac{\eta}{2}\right)\\ &\left(-\frac{15}{4}\operatorname{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)+\frac{15}{8}\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh^3\left(\frac{\eta}{2}\right)\right). \end{aligned} \tag{30}$$

The first approximation $\zeta_1(\eta,\tau)$ is given by eq. (9)

$$\begin{gathered} \frac{\partial\zeta_1(\eta,\tau)}{\partial\tau}+\mathrm{A}_1\left[\zeta_0(\eta,\tau)\right]N\left[\zeta_0(\eta,\tau)\right]+\mathrm{A}_2\left[\zeta_0(\eta,\tau),C_j\right]=0,\\ \zeta_1(\eta,0)=0, \end{gathered} \tag{31}$$

Here we again choose A_1 and A_2 similar as shown in above problem

$$\begin{cases} \mathrm{A}_1=C_1\left[\operatorname{sech}\left(\frac{\eta}{2}\right)\right]^2+C_2\left[\operatorname{sech}\left(\frac{\eta}{2}\right)\right]^4\\ \mathrm{A}_2=C_3\left[\operatorname{sech}\left(\frac{\eta}{2}\right)\right]^6+C_4\left[\operatorname{sech}\left(\frac{\eta}{2}\right)\right]^8. \end{cases} \tag{32}$$

Using eq. (29), and (30) into Eq. (31), we get the first approximation as

$$\begin{aligned} \zeta_1(\eta,\tau)=&\tau C_3\operatorname{sech}^6\left(\frac{\eta}{2}\right)-\tau C_4\operatorname{sech}^8\left(\frac{\eta}{2}\right)-14.0625\tau C_1\operatorname{sech}^6\left(\frac{\eta}{2}\right)\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)\\ &-14.0625\tau C_2\operatorname{sech}^6\left(\frac{\eta}{2}\right)\operatorname{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)-14.0625\tau C_1\operatorname{sech}^4\left(\frac{\eta}{2}\right)\operatorname{sech}^2\left(\frac{\eta}{2}\right)\\ &\tanh\left(\frac{\eta}{2}\right)-14.0625\tau C_2\operatorname{sech}^4\left(\frac{\eta}{2}\right)\operatorname{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right). \end{aligned} \tag{33}$$

Adding eq. (29) and eq. (33) we get 1st order approximate solution as

$$\tilde{\zeta}(\eta,\tau)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}\eta\right)+\tau C_3\operatorname{sech}^6\left(\frac{\eta}{2}\right)-\tau C_4\operatorname{sech}^8\left(\frac{\eta}{2}\right)-14.0625\tau C_1\operatorname{sech}^6\left(\frac{\eta}{2}\right)$$
$$\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)-14.0625\tau C_2\operatorname{sech}^6\left(\frac{\eta}{2}\right)\operatorname{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)-14.0625\tau C_1 \tag{34}$$
$$\operatorname{sech}^4\left(\frac{\eta}{2}\right)\operatorname{sech}^2\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right)-14.0625\tau C_2\operatorname{sech}^4\left(\frac{\eta}{2}\right)\operatorname{sech}^4\left(\frac{\eta}{2}\right)\tanh\left(\frac{\eta}{2}\right).$$

4. Numerical Results

We exhibit the correctness of our process for the provided equations and comparison with the VHPM for different values τ .

Result (1). To find the values of unknown parameters $C_i,\ i=1,2,3..$ we used the collection method.

$$C_1=0.4921659704908877,\ C_2=-0.2606759872371297,$$
$$C_3=-0.05781845184781508,\ C_4=-0.10563165777129231 \tag{35}$$

By substituting these values in Eq. (23), we obtained the first-order solution for the mCH equation.

Result (2). similarly, we used the collection method for finding values of $C_i,\ i=1,2,3..$ which are given as follow,

$$C_1=0.3875663632499093,\ C_2=-0.19320905798049545,$$
$$C_3=-0.46302224355411176,\ C_4=0.014099963170806097. \tag{36}$$

Using E. (36) eq. (34), we obtained the first-order solution for the DP equation.

Numerical values are tabulated for first-order OAFM and VHPM solution for the mCH equation in tables (1-2) at $\tau=0.01$ and $\tau=0.001$respectively. Tables (3-4) present the approximate solution of OAFM and VHPM solution for DP equation when $\tau=0.01$ and $\tau=0.001$.

Figures (1-3) present the 3D surfaces of first-order OAFM, VHPM, and exact solution for mCH equation respectively. Figures (4-6) show the 3D surfaces first order of OAFM, Exact solution, and VHPM solution for mDP equation respectively. Figures (7-8) show the comparison of 1st order OAFM, VHPM, and exact solution for mCH and mDP equation at $\tau=0.1$ respectively. From the tabulated results and graphical we conclude that the OAFM solution is very closed to the exact solution as compared to VHPM.

Table 1.

Numerical results were obtained by first-order OAFM and VHPM solution $\tau=0.01$for the solution of the mCH equation.

η	OAFM	Exact	Abs Error VHPM [12]	Abs Error OAFM
-1.0	-1.56446	-1.5583	0.0197059	0.00616137
-0.5	-1.87269	-1.87067	0.0166079	0.0020242
0.5	-1.88476	-1.88908	0.0169162	0.00432592
1.0	-1.57996	-1.58737	0.0198189	0.00741897

Table 2.
Numerical results were obtained by first order OAFM and VHPM solution $\tau = 0.001$ for the solution of the mCH equation.

η	OAFM	Exact	Abs Error VHPM [12]	Abs Error OAFM
-1.0	-1.57232	-1.57144	0.00197555	0.00061118
-0.5	-1.8793	-1.87911	0.00167455	0.000188656
0.5	-1.8805	-1.88095	0.00167764	0.000446577
1.0	-1.5736	-1.57435	0.00197668	0.000747114

Table 3.
Numerical results obtained by first-order OAFM and VHPM solution at $\tau = 0.01$ for the solution of the mDP equation.

η	OAFM	Exact	Abs Error VHPM [12]	Abs Error OAFM
-1.0	-1.46494	-1.45747	0.0230771	0.00747007
-0.5	-1.7529	-1.75151	0.019418	0.00138805
0.5	-1.76468	-1.77309	0.0198696	0.00840817
1.0	-1.47984	-1.49154	0.0232427	0.011701

Table 4.
Numerical results obtained by first order OAFM and VHPM solution at $\tau = 0.001$ for the solution of mDP equation.

η	OAFM	Exact	Abs Error VHPM [12]	Abs Error OAFM
-1.	-1.47362	-1.47289	0.00231493	0.000739793
-0.5	-1.76157	-1.76145	0.00196192	0.000118684
0.5	-1.76247	-1.7636	0.00196643	0.000861343
1.	-1.47511	-1.47629	0.00231658	0.00117779

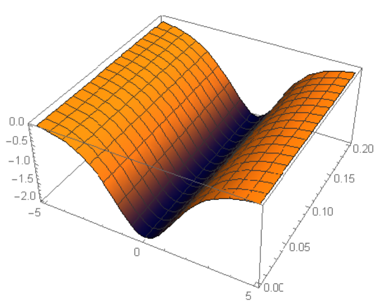

Figure 1: First order OAFM solution for mCH equation

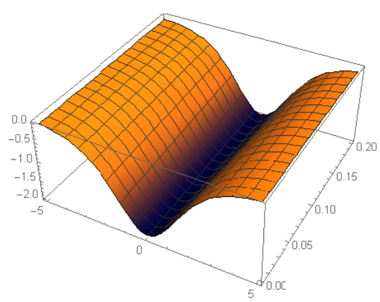

Figure 2: Exact solution for mCH equation

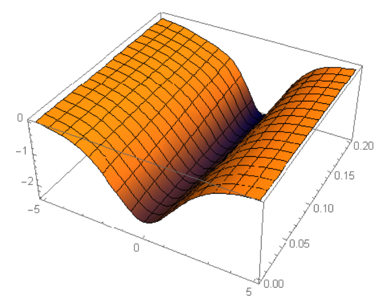

Figure 3: First order VHPM solution for mCH equation

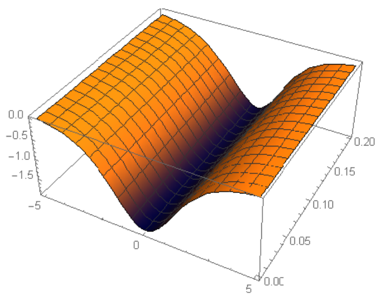

Figure 4: First order OAFM solution for mDP equation

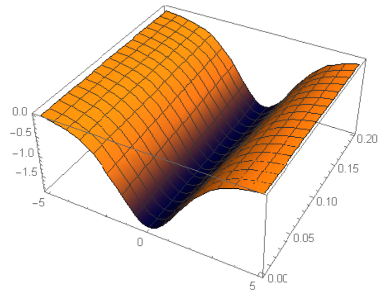

Figure 5: First order OAFM solution for mDP equation

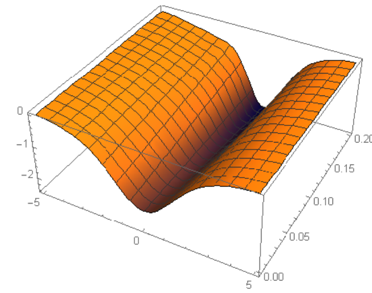

Figure 6: First order OAFM solution for mDP equation.

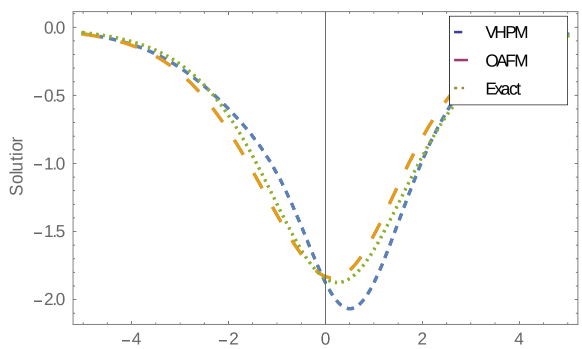

Figure 7: 2D surface are VHPM, OAFM and exact solution at τ = 0.1 for mCH equation.

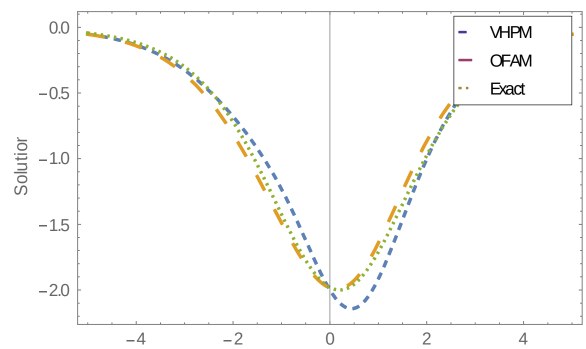

Figure 8: 2D surface are VHPM, OAFM and exact solution at τ = 0.1 for mDP equation.

5. Conclusion

In this work, the suggested method test for the approximate solution of mCH and mDP equations. The numerical results have been compared with VHPM. From the results, we conclude that OAFM converges rapidly than VHPM after only one iteration.

Competing Interests

The author(s) declare no competing interests.

References

[1] El-Sayed, A. M. A., and M. Gaber. The Adomian decomposition method for solving partial differential equations of fractal order in finite domains. *Physics Letters A.* **359** (3), 175-182 (2006).

[2] Dehghan, Mehdi, Jalil Manafian, and Abbas Saadatmandi. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numerical Methods for Partial Differential Equations: An International Journal*. **26** (2), 448-479 (2010).

[3] He, Ji-Huan. Homotopy perturbation method for solving boundary value problems. *Physics letters A*. **350** (1-2), 87-88 (2006).

[4] Nawaz, Rashid, et al. Optimum solutions of fractional order Zakharov–Kuznetsov equations. *Complexity*, (2019).

[4] He, Ji-Huan. Variational iteration method–a kind of non-linear analytical technique: some examples. *International journal of non-linear mechanics.* **34** (4), 699-708 (1999).

[5] Abdou, M. A., and A. A. Soliman. New applications of variational iteration method. *Physica D: Nonlinear Phenomena*. **211** (1-2), 1-8 (2005).

[6] Marinca, Bogdan, and Vasile Marinca. Approximate analytical solutions for thin film flow of a fourth grade fluid down a vertical cylinder. *Proceed Romanian Academy, Series A.* **19** (1), 69-76 (2018).

[7] Zada, L., Nawaz, R., Nisar, K. S., Tahir, M., Yavuz, M., Kaabar, M. K., & Martínez, F. New approximate-analytical solutions to partial differential equations via auxiliary function method. *Partial Differential Equations in Applied Mathematics*, **4**, 100045, (2021).

[8] Zada, Laiq, et al. New algorithm for the approximate solution of generalized seventh order Korteweg-Devries equation arising in shallow water waves. *Results in Physics.* **20**, 103744 (2021).

[9] Zada, L., et al. A New Approach for Solving Fredholm Integro-Differential Equations. *Information Sciences Letters.* **10** (3), 3 (2021).

[10] Zada, Laiq, et al. New approximate-analytical solutions to partial differential equations via auxiliary function method. *Partial Differential Equations in Applied Mathematics*. **4**, 100045 (2021).

[11] Wazwaz, Abdul-Majid. Solitary wave solutions for modified forms of Degasperis–Procesi and Camassa–Holm equations. *Physics Letters A.* **352** (6), 500-504 (2006).

[12] Yousif, Majeed A., Bewar A. Mahmood, and Fadhil H. Easif. A New Analytical Study of Modified Camassa-Holm and Degasperis-Procesi Equations. *American Journal of Computational Mathematics*, **5** (3), 267-273 (2015).

[13] Zhang, Ben-gong, Shao-yong Li, and Zheng-rong Liu. Homotopy perturbation method for modified Camassa–Holm and Degasperis–Procesi equations. *Physics Letters A.* **372** (11), 1867-1872 (2008).

[14] Behera, Ratikanta, and Mani Mehra. Approximate solution of modified Camassa-Holm and Degasperis-Procesi equations using wavelet optimized finite difference method. *International Journal of Wavelets, Multiresolution and Information Processing* **11** (02) (2013).